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Perturbation theory for the nearly integrable non-linear equations associated with a modified Zakharov–Shabat scattering problem

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Abstract. Recently it has been shown that the derivative non-linear Schrödinger equation, concerned with wave propagation in plasmas, can be associated with a modified Zakharov–Shabat inverse scattering problem. In this paper we produce an operator formula for the most general system of equations solvable by this method and develop a perturbation theory capable of determining the variation in the scattering data to first order. We illustrate the theory by applying it to the derivative non-linear Schrödinger equation containing an additional perturbing harmonic forcing term, and consider the effect of this perturbation on an algebraic soliton.

1. The scattering problem

We consider the following scattering problem,

$$V_x = PV \qquad \text{where } P(\zeta) = \begin{pmatrix} -\mathrm{i}\zeta^2 & q\zeta \\ r\zeta & \mathrm{i}\zeta^2 \end{pmatrix} \tag{1.1}$$

with $q, r \to 0$ as $|x| \to \infty$ and V a two-component column vector.

This scattering problem has recently been shown to be intimately related to the massive Thirring model in characteristic coordinates (Michaelov 1976, Morris 1979, Kaup and Newell 1977) and the derivative non-linear Schrödinger equation of plasma physics (Kaup and Newell 1978a). Following Newell and Kaup (Kaup 1976, Newell 1980) we develop a perturbation theory for the nearly solvable evolution equations determined by equation (1.1).

Define the fundamental matrix solutions Φ and Ψ of equation (1.1) by the asymptotic relationships

$$\Phi \sim \begin{pmatrix} \exp(-\mathrm{i}\zeta^2 x) & 0 \\ 0 & -\exp(\mathrm{i}\zeta^2 x) \end{pmatrix}, \qquad x \to -\infty, \tag{1.2}$$

$$\Psi \sim \begin{pmatrix} 0 & \exp(-\mathrm{i}\zeta^2 x) \\ \exp(\mathrm{i}\zeta^2 x) & 0 \end{pmatrix}, \qquad x \to +\infty. \tag{1.3}$$

The scattering matrix A is defined by

$$\Phi = \Psi A, \qquad A(\zeta) = \begin{pmatrix} b(\zeta) & -\bar{a}(\zeta) \\ a(\zeta) & \bar{b}(\zeta) \end{pmatrix}. \tag{1.4}$$

From equation (1.1) one easily shows that for an arbitrary variation $\delta\Phi$ of Φ we have

$$(\Phi^{-1}\delta\Phi)_x = \Phi^{-1}\delta P\Phi. \tag{1.5}$$

Integrating over $[-L, L]$ and allowing $L \to \infty$, we obtain

$$A^{-1}\delta A = \zeta\int_{-\infty}^{\infty}\begin{pmatrix} -\delta q\bar{\phi}_2\phi_2 + \delta r\phi_1\bar{\phi}_1, & -\delta q\bar{\phi}_2^2 + \delta r\bar{\phi}_1^2 \\ \delta q\phi_2^2 - \delta r\phi_1^2, & \delta q\bar{\phi}_2\phi_2 - \delta r\phi_1\bar{\phi}_1\end{pmatrix}\mathrm{d}x + A^{-1}A_\zeta\delta\zeta, \tag{1.6}$$

where ϕ, $\bar{\phi}$, ψ and $\bar{\psi}$ are defined by

$$\Phi = (\phi, \bar{\phi}) \qquad \Psi = (\psi, \bar{\psi}) \tag{1.7}$$

and we have used equation (1.5) in the case of a variation of ζ above.

If we define

$$I(u, v) = \int_{-\infty}^{\infty}(-\delta q u_2 v_2 + \delta r u_1 v_1)\,\mathrm{d}x = \int_{-\infty}^{\infty}\begin{pmatrix}\delta r \\ -\delta q\end{pmatrix}\cdot\begin{pmatrix}u_1 v_1 \\ u_2 v_2\end{pmatrix}\mathrm{d}x, \tag{1.8}$$

we have

$$\delta a = -\zeta I(\phi, \psi) + \delta\zeta a_\zeta, \tag{1.9}$$

$$\delta b = \zeta I(\phi, \bar{\psi}) + \delta\zeta b_\zeta, \tag{1.10}$$

$$\delta\bar{a} = -\zeta I(\bar{\phi}, \bar{\psi}) + \delta\zeta\bar{a}_\zeta, \tag{1.11}$$

$$\delta\bar{b} = -\zeta I(\bar{\phi}, \psi) + \delta\zeta\bar{b}_\zeta, \tag{1.12}$$

when the potentials q and r are on compact support. In this case Ablowitz *et al* (1974) and Kaup and Newell (1978a, b) have shown that the fundamental matrix solutions and the scattering matrix are analytic in the whole ζ plane. We shall impose this condition for the remainder of this section.

Asymptotically we have

$$\Phi\begin{pmatrix}\exp(\mathrm{i}\zeta^2 x) & 0 \\ 0 & \exp(-\mathrm{i}\zeta^2 x)\end{pmatrix} \sim \begin{pmatrix}\exp(\mathrm{i}\mu^-(x)) & 0 \\ 0 & -\exp(-\mathrm{i}\mu^-(x))\end{pmatrix} + \mathrm{o}(1), \qquad |\zeta| \to \infty, \tag{1.13}$$

$$\Psi\begin{pmatrix}\exp(-\mathrm{i}\zeta^2 x) & 0 \\ 0 & \exp(\mathrm{i}\zeta^2 x)\end{pmatrix} \sim \begin{pmatrix}0 & \exp(-\mathrm{i}\mu^+(x)) \\ \exp(\mathrm{i}\mu^+(x)) & 0\end{pmatrix} + \mathrm{o}(1), \qquad |\zeta| \to \infty, \tag{1.14}$$

$$A \sim \begin{pmatrix}0 & -\mathrm{e}^{-\mathrm{i}\mu} \\ \mathrm{e}^{\mathrm{i}\mu} & 0\end{pmatrix} + \mathrm{o}(1), \qquad |\zeta| \to \infty, \tag{1.15}$$

where

$$\mu^-(x) = \tfrac{1}{2}\int_{-\infty}^{x} rq\,\mathrm{d}x, \qquad \mu^+(x) = \tfrac{1}{2}\int_{x}^{\infty} rq\,\mathrm{d}x \qquad \text{and } \mu = \mu^-(x) + \mu^+(x). \tag{1.16}$$

Relations (1.9)–(1.12) and (1.13)–(1.15) imply

$$a\bar{a} + b\bar{b} = 1. \tag{1.17}$$

As shown by Kaup and Newell (1978a, b), $a(\zeta)$ and $\bar{a}(\zeta)$ are even functions and $b(\zeta)$, $\bar{b}(\zeta)$ are odd functions of ζ. Thus the zeros of $a(\zeta)$ and $\bar{a}(\zeta)$, which are the bound-state eigenvalues of equation (1.1), arise in pairs $(\zeta, -\zeta)$ and at such a zero, which we assume to be simple, we have

$$\phi(\zeta_{2k}) = b_{2k}\psi(\zeta_{2k}), \qquad \bar{\phi}(\zeta_{2j}) = \bar{b}_{2j}\bar{\psi}(\zeta_{2j}). \tag{1.18}$$

We have introduced the convention that if ζ_{2k}, $(\bar{\zeta}_{2j})$ is an eigenvalue in the first (fourth) quadrant, then $\zeta_{2k+1} = -\zeta_{2k}$, $(\bar{\zeta}_{2j+1} = -\bar{\zeta}_{2j})$ is an eigenvalue in the third (second) quadrant, $k = 1, \ldots, M$, $(j = 1, \ldots, N)$. It is convenient at this point to introduce the normalisation constants for the bound states,

$$C_k = \frac{-\mathrm{i}b_{2k}}{\zeta_{2k}a'_k} = \frac{-\mathrm{i}b_{2k+1}}{\zeta_{2k+1}a'_k} \qquad \bar{C}_j = \frac{\mathrm{i}\bar{b}_{2j}}{\bar{\zeta}_{2j}\bar{a}'_j} = \frac{\mathrm{i}\bar{b}_{2j+1}}{\bar{\zeta}_{2j+1}\bar{a}'_j},$$

where

$$a'_k = (\partial a/\partial\lambda)|_{\lambda=\lambda_k} \qquad \text{and} \qquad \lambda = \zeta^2, \qquad \lambda_k = \zeta_{2k}^2 = \zeta_{2k+1}^2. \tag{1.19}$$

From equation (1.4) it then follows, upon integrating around the contours in figure 1, that

$$\Psi\begin{pmatrix} \exp(-\mathrm{i}\zeta^2 x) & 0 \\ 0 & \exp(\mathrm{i}\zeta^2 x) \end{pmatrix} = \begin{pmatrix} 0 & \exp(-\mathrm{i}\mu^+(x)) \\ \exp(\mathrm{i}\mu^+(x)) & 0 \end{pmatrix}$$
$$+\frac{1}{2\pi\mathrm{i}}\begin{pmatrix} 0 & \displaystyle\int_C \frac{\mathrm{d}\zeta'}{(\zeta-\zeta')}\frac{b}{a}(\zeta')\Psi(\zeta')\exp(\mathrm{i}\zeta'^2 x) \\ \displaystyle\int_{\bar{C}} \frac{\mathrm{d}\zeta}{(\zeta-\zeta')}\frac{\bar{b}}{\bar{a}}(\zeta')\bar{\Psi}(\zeta')\exp(-\mathrm{i}\zeta'^2 x) & 0 \end{pmatrix}. \tag{1.20}$$

We postulate the existence of the transformation operator

$$\Psi = \begin{pmatrix} \exp(-\mathrm{i}\mu^+(x)) & 0 \\ 0 & \exp(\mathrm{i}\mu^+(x)) \end{pmatrix}\left[\begin{pmatrix} 0 & \exp(-\mathrm{i}\zeta^2 x) \\ \exp(\mathrm{i}\zeta^2 x) & 0 \end{pmatrix}\right.$$
$$\left.+\int_x^\infty \begin{pmatrix} \zeta K_1(x,s), & \bar{K}_1(x,s) \\ K_2(x,s), & \zeta\bar{K}_2(x,s) \end{pmatrix}\begin{pmatrix} \exp(\mathrm{i}\zeta^2 s) & 0 \\ 0 & \exp(-\mathrm{i}\zeta^2 s) \end{pmatrix}\mathrm{d}s\right]. \tag{1.21}$$

Defining

$$K = \begin{pmatrix} K_1 & \bar{K}_1 \\ K_2 & \bar{K}_2 \end{pmatrix},$$

equation (1.21) satisfies equation (1.1) provided

$$K(x,s)|_{s\to\infty} = 0, \tag{1.22}$$

$$q(x)\exp(2\mathrm{i}\mu^+(x)) = -2K_1(x,x), \qquad r(x)\exp(-2\mathrm{i}\mu^+(x)) = -2\bar{K}_2(x,x), \tag{1.23}$$

$$\begin{aligned}&(\partial_x+\partial_s)L(x,s) = -\tfrac{1}{2}\mathrm{i}K_1(x,s)\exp(-\mathrm{i}\mu^+(x))\partial_x[r\exp(-\mathrm{i}\mu^+(x))],\\ &L \equiv K_2 - \tfrac{1}{2}rK_1\exp(-2\mathrm{i}\mu^+(x)),\end{aligned} \tag{1.24}$$

$$\begin{aligned}&(\partial_x+\partial_s)\bar{L}(x,s) = \tfrac{1}{2}\mathrm{i}\bar{K}_2(x,s)\exp(\mathrm{i}\mu^+(x))\partial_x[q\exp(\mathrm{i}\mu^+(x))],\\ &\bar{L} \equiv \bar{K}_1 + \tfrac{1}{2}q\bar{K}_2\exp(2\mathrm{i}\mu^+(x)),\end{aligned} \tag{1.25}$$

$$\begin{aligned}&(\partial_x-\partial_s)K_1(x,s) = q(x)L(x,s)\exp(2\mathrm{i}\mu^+(x)),\\ &(\partial_x-\partial_s)\bar{K}_2(x,s) = r(x)\bar{L}(x,s)\exp(-2\mathrm{i}\mu^+(x)).\end{aligned} \tag{1.26}$$

Characteristic theory and uniqueness and existence theory of ordinary differential equations then ensure that equation (1.21) is valid. Substituting equation (1.21) into equation (1.20) and taking Fourier transforms along the contour R in figure 1 we obtain the Marchenko equations

$$K(x,y)+\begin{pmatrix}-\bar{F}(x+y) & 0\\ 0 & F(x+y)\end{pmatrix} + \int_x^\infty \begin{pmatrix}-\bar{K}_1(x,s)\bar{F}(s+y), & -\mathrm{i}K_1(x,s)F'(s+y)\\ -\mathrm{i}\bar{K}_2(x,s)\bar{F}'(s+y), & K_2(x,s)F(s+y)\end{pmatrix}\mathrm{d}s=0 \tag{1.27}$$

where

$$F(z)=\frac{1}{2\pi}\int_C \frac{b}{a}(\zeta)\exp(\mathrm{i}\zeta^2 z)\,\mathrm{d}\zeta, \qquad \bar{F}(z)=\frac{1}{2\pi}\int_{\bar{C}} \frac{\bar{b}}{\bar{a}}(\zeta)\exp(-\mathrm{i}\zeta^2 z)\,\mathrm{d}\zeta$$

and

$$F'(z)=\mathrm{d}F(z)/\mathrm{d}z. \tag{1.28}$$

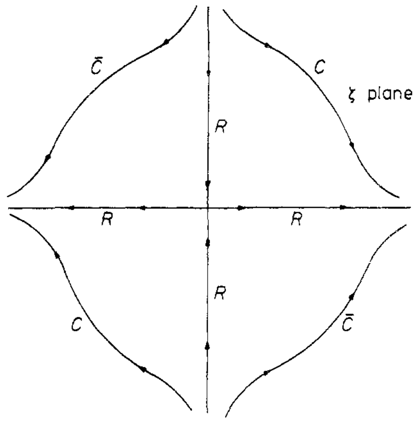

Figure 1. The contours C, $\bar{C}$, and R in the complex ζ plane.

Define

$$\rho(\lambda)=\frac{b}{a}(\zeta)\zeta^{-1} \qquad \text{and} \qquad \bar{\rho}(\lambda)=\frac{\bar{b}}{\bar{a}}(\zeta)\zeta^{-1} \tag{1.29}$$

and assume a, $\bar{a}$ have only simple zeros; then the definitions (1.28) for non-compact support become

$$F(z)=\sum_{k=1}^{M} C_k\exp(\mathrm{i}\lambda_k z)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(\lambda)\exp(\mathrm{i}\lambda z)\,\mathrm{d}\lambda,$$

$$\bar{F}(z)=\sum_{j=1}^{N} \bar{C}_j\exp(-\mathrm{i}\bar{\lambda}_j z)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\rho}(\lambda)\exp(-\mathrm{i}\lambda z)\,\mathrm{d}\lambda. \tag{1.30}$$

N-soliton solutions are obtained by solving equation (1.27) with $\rho=0$, $\bar{\rho}=0$ in equation (1.28). In particular the one-soliton solutions are found by taking

$$F(z)=C\exp(\mathrm{i}\lambda z), \qquad \bar{F}(z)=\bar{C}\exp(-\mathrm{i}\bar{\lambda}z); \tag{1.31}$$

then from equations (1.23) and (1.27) we have

$$\begin{pmatrix} K_1(x, y) \\ K_2(x, y) \end{pmatrix} = \bar{C} \exp[-i\bar{\lambda}(x+y)] \begin{pmatrix} D^{-1}(x) \\ [-iC\bar{\lambda}/(\lambda-\bar{\lambda})] \exp(2i\lambda x)\bar{D}^{-1}(x) \end{pmatrix}, \tag{1.32}$$

$$\begin{pmatrix} \bar{K}_1(x, y) \\ \bar{K}_2(x, y) \end{pmatrix} = -C \exp[i\lambda(x+y)] \begin{pmatrix} [i\bar{C}\lambda/(\lambda-\bar{\lambda})] \exp(-2i\bar{\lambda}x) D^{-1}(x) \\ \bar{D}^{-1}(x) \end{pmatrix}, \tag{1.33}$$

with

$$D(x) = \left(1 - \frac{\lambda C\bar{C} \exp 2i(\lambda-\bar{\lambda})x}{(\lambda-\bar{\lambda})^2}\right), \qquad \bar{D}(x) = \left(1 - \frac{\bar{\lambda} C\bar{C} \exp 2i(\lambda-\bar{\lambda})x}{(\lambda-\bar{\lambda})^2}\right). \tag{1.34}$$

Equations (1.23) and (1.33) define the one-soliton solutions

$$q(x) = -2\bar{C} \exp(-2i\bar{\lambda}x) D(x)\bar{D}^{-2}(x), \tag{1.35}$$

$$r(x) = 2C \exp(2i\lambda x)\bar{D}(x) D^{-2}(x), \tag{1.36}$$

$$\exp i(\mu^{+}(x)) = \bar{D}(x) D^{-1}(x). \tag{1.37}$$

Introducing the definitions

$$z = -i\, C \exp 2i\, \lambda x/(\lambda-\bar{\lambda}), \qquad \bar{z} = i\, \bar{C} \exp -2i\bar{\lambda}x/(\lambda-\bar{\lambda}), \tag{1.38}$$

equation (1.21) yields for the corresponding bound-state eigenfunctions

$$\psi(x, \lambda) = e^{i\lambda x}\left(D^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda^{1/2}\bar{z}\bar{D}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \tag{1.39}$$

$$\bar{\psi}(x, \bar{\lambda}) = e^{-i\bar{\lambda}x}\left(\bar{D}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \bar{\lambda}^{1/2} z D^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right). \tag{1.40}$$

For an arbitrary point on the real line, $p \in R$, we have

$$\psi(x, p) = \begin{pmatrix} \dfrac{i\bar{C}p^{1/2}\, e^{[i(p-2\bar{\lambda})x]}}{\bar{D}(p-\bar{\lambda})} \\ \dfrac{e^{ipx}\{-\bar{\lambda}C\bar{C}\, e^{[2i(\lambda-\bar{\lambda})x]}(p-\lambda) + (p-\bar{\lambda})(\lambda-\bar{\lambda})^2\}}{D(\lambda-\bar{\lambda})^2(p-\bar{\lambda})} \end{pmatrix}. \tag{1.41}$$

2. Solvable equations and perturbation theory for the inverse scattering problem (1.1)

First notice that in the case of the exactly solvable equations, (1.9)–(1.12) become, when the variation is with respect to time,

$$\lambda_{kt} = 0, \qquad \bar{\lambda}_{jt} = 0, \tag{2.1}$$

$$a_t = -\zeta I_t(\phi, \psi), \tag{2.2}$$

$$\bar{a}_t = -\zeta I_t(\bar{\phi}, \bar{\psi}), \tag{2.3}$$

$$b_t = \zeta I_t(\phi, \bar{\psi}), \tag{2.4}$$

$$\bar{b}_t = -\zeta I_t(\bar{\phi}, \psi), \tag{2.5}$$

where

$$I_t(u, v) = \int_{-\infty}^{\infty} (r_t u_1 v_1 - q_t u_2 v_2)\, \mathrm{d}x. \tag{2.6}$$

When the potentials are on compact support, these can be written using equation (1.4) as

$$\rho_t(\lambda) = (1/a^2(\lambda)) I_t(\phi, \phi), \tag{2.7}$$

$$\bar{\rho}_t(\lambda) = (1/\bar{a}^2(\lambda)) I_t(\bar{\phi}, \bar{\phi}). \tag{2.8}$$

If we *define*

$$I_t(\phi, \phi) = ab\zeta^{-1}\Omega(\lambda), \tag{2.9}$$

$$I_t(\bar{\phi}, \bar{\phi}) = -\bar{a}\bar{b}\zeta^{-1}\bar{\Omega}(\lambda), \tag{2.10}$$

then equations (2.7) and (2.8) become

$$\rho_t(\lambda) = \Omega(\lambda)\rho(\lambda) \qquad \text{and} \qquad \bar{\rho}_t(\lambda) = -\bar{\Omega}(\lambda)\bar{\rho}(\lambda), \tag{2.11}$$

which are integrable. Minor generalisations are achieved by allowing the variation to be a directional derivative which includes other spatial variables (e.g. $\delta \equiv \partial/\partial t + \boldsymbol{h}(t, \boldsymbol{y}) . \partial/\partial \boldsymbol{y}$). From equation (1.1) we also have the relation

$$ab = \phi_1\phi_2|_{-\infty}^{\infty} = \zeta \int_{-\infty}^{\infty} (q\phi_2^2 + r\phi_1^2)\, \mathrm{d}x, \tag{2.12}$$

and consequently equation (2.9) can be written as

$$\int_{-\infty}^{\infty} [(r_t - r\Omega(\lambda))\phi_1^2 - (q_t + q\Omega(\lambda))\phi_1^2]\, \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \begin{pmatrix} r_t - r\Omega(\lambda) \\ -(q_t + q\Omega(\lambda)) \end{pmatrix} \cdot \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix} \mathrm{d}x = 0. \tag{2.13}$$

Equation (2.13) involves an inner product between the 'squared' eigenfunctions. Using equation (1.1) one obtains

$$\partial_x \phi^{(2)} = \lambda \begin{pmatrix} -2\mathrm{i} + 2qI^- r, & 2qI^- q \\ 2rI^- r, & 2\mathrm{i} + 2rI^- q \end{pmatrix} \phi^{(2)}, \tag{2.14}$$

where

$$\phi^{(2)} = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}$$

and

$$(I\bar{r})v(x) = \int_{-\infty}^{x} r(y)v(y)\, \mathrm{d}y, \qquad (I^+ r)v(x) = \int_{x}^{\infty} r(y)v(y)\, \mathrm{d}y. \tag{2.15}$$

It follows that

$$L\phi^{(2)} = \lambda\phi^{(2)} \tag{2.16}$$

and

$$L = \frac{1}{2\mathrm{i}} \begin{pmatrix} -\partial_x + \mathrm{i}qI^-(r\partial_x), & -\mathrm{i}qI^-(q\partial_x) \\ -\mathrm{i}rI^-(r\partial_x), & \partial_x + \mathrm{i}rI^-(q\partial_x) \end{pmatrix}. \tag{2.17}$$

The adjoint operator is defined by

$$L^{\mathrm{A}} = \frac{1}{2\mathrm{i}}\partial_x\begin{pmatrix} 1-\mathrm{i}(rI^+q), & \mathrm{i}(rI^+r) \\ \mathrm{i}(qI^+q), & -1+\mathrm{i}(qI^+r)\end{pmatrix}. \tag{2.18}$$

The solvable equations are therefore, from equation (2.13),

$$\begin{pmatrix} r_t \\ -q_t \end{pmatrix} = \Omega(L^{\mathrm{A}})\begin{pmatrix} r \\ q \end{pmatrix}, \tag{2.19}$$

provided $\bar{\Omega} = \Omega$. In this paper we assume this to be the case, although presumably removing this restriction provides other interesting evolution equations.

As examples of the general evolution equations solvable by equation (1.1) we consider cases when Ω is an entire function of λ and one example when Ω is a singular function of λ.

Example 1. $\Omega = 2\mathrm{i}\lambda$

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = \begin{pmatrix} r_x \\ q_x \end{pmatrix} \tag{2.20}$$

is just a linear wave equation.

Example 2. $\Omega = 4\mathrm{i}\lambda^2$

$$\begin{pmatrix} r_t \\ -q_t \end{pmatrix} = \begin{pmatrix} -\mathrm{i}r_{xx} + (r^2q)_x \\ -\mathrm{i}q_{xx} - (q^2r)_x \end{pmatrix}. \tag{2.21}$$

When $r = \pm q^*$ the system reduces to a single equation, the derivative non-linear Schrödinger equation,

$$\mathrm{i}q_t = -q_{xx} \pm \mathrm{i}(q^2q^*)_x \tag{2.22}$$

which governs the propagation of circularly polarised non-linear Alfven waves in plasmas (Mio *et al* 1976, Mjølhus 1976, Ichikawa and Watanabe 1977).

Example 3. $\Omega = -8\mathrm{i}\lambda^3$

$$\begin{matrix} r_t \\ -q_t \end{matrix} = \begin{matrix} (r_{xx} + 3\mathrm{i}rqr_x - q^2r^3)_x, \\ -(q_{xx} - 3\mathrm{i}qrq_x - q^3r^2)_x. \end{matrix} \tag{2.23}$$

When $r = \pm \mathrm{i}q$ this yields an integrable model for long lattice waves,

$$q_t = (q_{xx} \pm 3q^2q_x + \tfrac{3}{2}q^5)_x. \tag{2.24}$$

Example 4. $\Omega = 1/2\mathrm{i}\lambda$

$$U_{xt} - \mathrm{i}UVU_x = U, \qquad V_{xt} + \mathrm{i}UVV_x = V,$$

with

$$U = r\exp(-2\mathrm{i}\mu^+) \qquad \text{and} \qquad V = q\exp(2\mathrm{i}\mu^+). \tag{2.25}$$

It follows from equations (2.18) and (2.19) in a similar fashion to the Zakharov–Shabat system (Ablowitz *et al* 1974) that the dispersion relation of the linearised equation determines the Ω for the non-linear equation. Thus for $q, r \ll 1$, we have from equations (1.23), (1.27) and (1.30)

$$q=\frac{1}{\pi}\int_{-\infty}^{\infty}\bar{\rho}\exp[-\mathrm{i}(2\lambda x+\mathrm{i}\Omega t)]\,\mathrm{d}\lambda, \qquad r=\frac{1}{\pi}\int_{-\infty}^{\infty}\rho\exp[\mathrm{i}(2\lambda x-\mathrm{i}\Omega t)]\,\mathrm{d}\lambda. \tag{2.26}$$

From equations (1.9)–(1.12) it is possible to obtain the variation of the scattering data, which we take to be the set $S=(\rho, \bar{\rho}, \lambda_k, \bar{\lambda}_j, C_k, \bar{C}_j, k=1, \ldots, M, j=1, \ldots, N)$, with respect to time when the potentials are not on compact support and do not necessarily correspond to one of the set of exactly solvable equations (2.19).

$$\rho_t(\lambda)=(1/a^2(\lambda))I_t(\phi, \phi), \tag{2.27}$$

$$\bar{\rho}_t(\lambda)=(1/\bar{a}^2(\lambda))I_t(\bar{\phi}, \bar{\phi}), \tag{2.28}$$

$$\mathrm{i}\lambda_{kt}=(1/C_k a_k'^2)I_t(\phi, \phi)_{\lambda=\lambda_k}, \tag{2.29}$$

$$\mathrm{i}\bar{\lambda}_{jt}=-(1/\bar{C}_j\bar{a}_j'^2)I_t(\bar{\phi}, \bar{\phi})_{\lambda=\bar{\lambda}_j}, \tag{2.30}$$

$$\begin{aligned}C_{kt}&=\frac{-\mathrm{i}}{a_k'^2}\left(\frac{\partial}{\partial\lambda}I_t(\phi, \phi)-I_t(\phi, \phi)\frac{a_k''}{a_k'}\right)\bigg|_{\lambda=\lambda_k}\\&=\lambda_k C_k\left(\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\bar{\rho}I_t(\bar{\psi}, \bar{\psi})+\rho I_t(\psi, \psi)}{(\lambda'-\lambda_k)}\,\mathrm{d}\lambda'-\mathrm{i}C_k\frac{\partial}{\partial\lambda}I_t(\psi, \psi)\bigg|_{\lambda=\lambda_k}\right.\\&\quad\left.-2\mathrm{i}\sum_{j\neq k}^{M}\frac{C_jI_t(\psi, \psi)}{(\lambda_j-\lambda_k)}\bigg|_{\lambda=\lambda_j}-2\mathrm{i}\sum_{s}^{N}\frac{\bar{C}_sI_t(\bar{\psi}, \bar{\psi})}{(\bar{\lambda}_s-\lambda_k)}\bigg|_{\lambda=\bar{\lambda}_s}+\frac{\mathrm{i}C_kI_t(\psi, \psi)}{\lambda_k}\bigg|_{\lambda=\lambda_k}\right),\end{aligned} \tag{2.31}$$

$$\begin{aligned}\bar{C}_{jt}&=\frac{\mathrm{i}}{\bar{a}_j'^2}\left(\frac{\partial}{\partial\lambda}I_t(\bar{\phi}, \bar{\phi})-I_t(\bar{\phi}, \bar{\phi})\frac{\bar{a}_j''}{\bar{a}_j'}\right)\bigg|_{\lambda=\bar{\lambda}_j}\\&=\bar{\lambda}_j\bar{C}_j\left(-\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\bar{\rho}I_t(\bar{\psi}, \bar{\psi})+\rho I_t(\psi, \psi)}{(\lambda'-\bar{\lambda}_j)}\,\mathrm{d}\lambda'+\mathrm{i}\bar{C}_j\frac{\partial}{\partial\lambda}I_t(\bar{\psi}, \bar{\psi})\bigg|_{\lambda=\bar{\lambda}_j}\right.\\&\quad\left.+2\mathrm{i}\sum_{k}^{M}\frac{C_kI_t(\psi, \psi)}{(\lambda_k-\bar{\lambda}_j)}\bigg|_{\lambda=\lambda_k}+2\mathrm{i}\sum_{s\neq j}^{N}\frac{\bar{C}_sI_t(\bar{\psi}, \bar{\psi})}{(\bar{\lambda}_s-\bar{\lambda}_j)}\bigg|_{\lambda=\bar{\lambda}_s}-\frac{\mathrm{i}\bar{C}_jI_t(\bar{\psi}, \bar{\psi})}{\bar{\lambda}_j}\bigg|_{\lambda=\bar{\lambda}_j}\right).\end{aligned} \tag{2.32}$$

We now, as an example, develop a perturbation theory for the effects of weak perturbations on solitons and apply it to the derivative non-linear Schrödinger equation for the case of a perturbed algebraic soliton. The almost integrable equations are, from equation (2.19),

$$\begin{pmatrix} r_t \\ -q_t \end{pmatrix}=\Omega(L^{\mathrm{A}})\begin{pmatrix} r \\ q \end{pmatrix}+\delta\begin{pmatrix} f(r, q) \\ g(r, q) \end{pmatrix}, \tag{2.33}$$

where δ is a small parameter and f, g are functionals of r, q and their x-derivatives.

For $r=\epsilon q^*$, $\epsilon=\pm 1$, the barred and unbarred quantities defined in § 1 are related in the following manner:

$$\bar{\psi}(\bar{\zeta})=M\psi^*(\zeta^*), \qquad \bar{\phi}(\bar{\zeta})=-\epsilon M\phi^*(\zeta^*), \qquad \bar{a}(\bar{\zeta})=a^*(\zeta^*),$$

$$\bar{b}(\bar{\zeta})=-\epsilon b^*(\zeta^*), \qquad \bar{C}_j=-\epsilon C_j^*, \qquad \bar{\lambda}_j=\lambda_j^*,$$

where

$$M=\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}. \tag{2.34}$$

This leads to the following relations between the kernels and associated functions occuring in the Marchenko equations (1.22)–(1.28):

$$\bar{F} = -\epsilon F^*, \qquad \bar{D} = D^*, \qquad \bar{L} = L^*,$$
$$\begin{pmatrix} \bar{K}_1 \\ \bar{K}_2 \end{pmatrix} = M \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix}, \tag{2.35}$$

which agrees with the work of Kaup and Newell (1978a, b). For the derivative non-linear Schrödinger equation, $\Omega = 4\mathrm{i}\lambda^2$ and equation (2.33) reduces to the single equation

$$q_t = \mathrm{i}q_{xx} + \epsilon(q^2 q^*)_x - \delta g(q, q^*). \tag{2.36}$$

The effects of this perturbation on the scattering data to first order in δ for a single soliton are easily obtained from equations (2.27)–(2.32):

$$\rho_t = \Omega(\lambda)\rho + \delta(1/a^2) I_p(\phi, \phi) + \mathrm{O}(\delta^2), \tag{2.37}$$

$$\mathrm{i}\lambda_t = (\delta/Ca'^2) I_\lambda(\phi, \phi) = -\delta C\lambda I_\lambda(\psi, \psi) + \mathrm{O}(\delta^2), \tag{2.38}$$

$$\begin{aligned} C_t &= \Omega(\lambda)C - \frac{\delta \mathrm{i}}{a'^2}\left(\frac{\partial J}{\partial \lambda'}(\phi, \phi)_{\lambda'=\lambda} - J(\phi, \phi)\frac{a''}{a'}\right) + \mathrm{O}(\delta^2) \\ &= \Omega(\lambda)C + \delta\lambda C\left(\frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\rho^* J^*(\psi, \psi) + \rho J(\psi, \psi)}{(\lambda' - \lambda)}\,\mathrm{d}\lambda' \right. \\ &\quad \left. - \mathrm{i}C\frac{\partial J}{\partial \lambda'}(\psi, \psi)_{\lambda'=\lambda} - \frac{2\mathrm{i}C^* J^*(\psi, \psi)}{(\lambda^* - \lambda)} + \frac{\mathrm{i}CJ}{\lambda}(\psi, \psi)\right) + \mathrm{O}(\delta^2), \end{aligned} \tag{2.39}$$

with

$$J(u, v) = \int_{-\infty}^{\infty} (g u_2 v_2 - \epsilon g^* u_1 v_1)\,\mathrm{d}x. \tag{2.40}$$

For an algebraic soliton, after rescaling the Jost functions, equations (1.39) and (1.43) become

$$\psi(x, \xi) = \begin{pmatrix} \dfrac{-\epsilon\xi^{1/2}\exp[-\mathrm{i}(\xi x + 2\sigma_0)]}{\Delta[\mathrm{i}\epsilon + 4\Delta^2(x - x_0)]} \\ \dfrac{\mathrm{e}^{\mathrm{i}\xi x}}{[-\mathrm{i}\epsilon + 4\Delta^2(x - x_0)]} \end{pmatrix} \tag{2.41}$$

$$\psi(x, p) = \begin{pmatrix} \dfrac{-\epsilon p^{1/2}\exp\{\mathrm{i}[(p - 2\xi)x - 2\sigma_0]\}}{\Delta[\mathrm{i}\epsilon + 4\Delta^2(x - x_0)]} \\ \dfrac{\epsilon\,\mathrm{e}^{\mathrm{i}px}[\mathrm{i}(p + \xi) + 4\xi(p - \xi)(x - x_0)]}{2\mathrm{i}\Delta^2[-\mathrm{i}\epsilon + 4\Delta^2(x - x_0)]} \end{pmatrix} \tag{2.42}$$

where $p \in R$. These representations are obtained by defining

$$C = 2\eta\Delta^{-1}\exp 2(\mathrm{i}\sigma_0 + \eta x_0), \qquad \lambda = \xi + \mathrm{i}\eta = \Delta^2(\epsilon + \mathrm{i}\beta) \tag{2.43}$$

and inserting them into equations (1.39) and (1.43), after rescaling by $\psi(x, p) \to (1/2\mathrm{i}\Delta^2)(p - \lambda^*)\psi(x, p)$ and taking the limit $\beta \to 0$. The function defined by equation (2.42) no longer satisfies the boundary condition (1.3) because of the rescaling. However even in this case we still obtain the equation (1.6) giving the variation of the scattering data for the *rescaled system*. Thus equations (2.37)–(2.40) are still valid, but the scattering data and the Jost functions are now those associated with the rescaled system.

The algebraic soliton is obtained from equation (1.35) upon using the representation (2.43) for C and taking the limit $\beta \to 0$ (Kaup and Newell 1978a, b). It can also be derived directly from either (2.41) or (2.42) and equation (1.1), and is given by

$$q = 4\Delta\epsilon \exp[-2\mathrm{i}(\epsilon\Delta^2 x + \sigma_0)] \frac{[-\mathrm{i}\epsilon + 4\Delta^2(x - x_0)]}{[\mathrm{i}\epsilon + 4\Delta^2(x - x_0)]^2}. \tag{2.44}$$

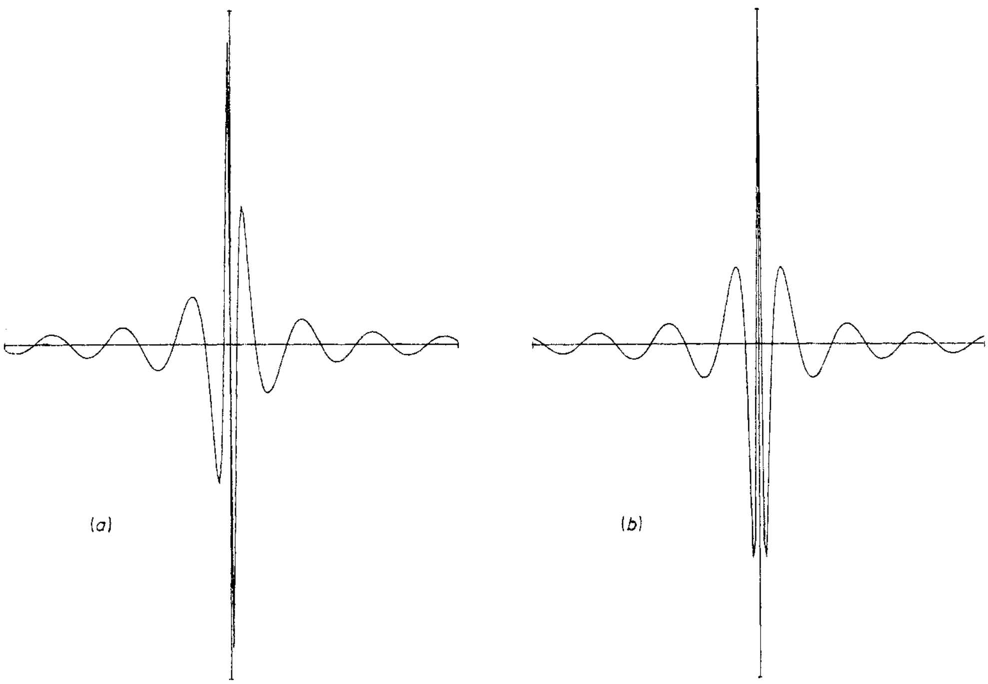

Figure 2. The algebraic soliton. The algebraic soliton depicted here has initial values $x_0 = 0$, $\sigma_0 = 0$ and $\Delta = 1$. The figure displays (a) the real and (b) the imaginary parts of the soliton, displayed as an envelope modulating an oscillatory wavetrain represented by $q = |q| \exp \mathrm{i}\theta$, where $|q| = 4\Delta/[1 + 16\Delta^4(x - x_0)^2]$, $\theta = -2(\epsilon\Delta^2 x + \sigma_0) + 3\alpha$,

$$\alpha = \cos^{-1}\left(\frac{4\Delta^2(x - x_0)}{[1 + 16\Delta^4(x - x_0)^2]^{1/2}}\right).$$

Mjølhus (1978) has shown that algebraic solitons arise in the transition from modulational stability to instability of an initial circularly polarised long Alfvén wave solution to the derivative non-linear Schrödinger equation. For the algebraic soliton with $g = |g(t)| \, \mathrm{e}^{2\mathrm{i}\omega t}$ the integrals in equations (2.37)–(2.39) can easily be evaluated:

$$J(\psi, \psi)_{\lambda=\xi} = (-\mathrm{i}\pi/2\xi) \exp[-(2\mathrm{i}\sigma_0 + \tfrac{1}{2})]|g(t)| \sin\gamma, \tag{2.45}$$

$$\frac{\partial}{\partial\lambda} J(\psi, \psi)_{\lambda=\xi} = \frac{-\mathrm{i}\pi \exp[-(2\mathrm{i}\sigma_0 + \tfrac{1}{2})]}{4\xi^2} |g(t)|(1 + 4\mathrm{i}\xi x_0) \sin\gamma, \tag{2.46}$$

$$J(\psi,\psi)_{\lambda=p}=(\pi p/4\xi^4)\,\mathrm{e}^{-1/2}\exp\{2\mathrm{i}[(p-\xi)x_0-\sigma_0]\}|g(t)|\{p(\xi-p)\cos(\gamma+\mathrm{i}\alpha)+\mathrm{i}[4\xi(p-\xi)-p(p+\xi)]\sin(\gamma+\mathrm{i}\alpha)\}, \quad (2.47)$$

with

$$\alpha=(p-\xi)/2\xi, \qquad \gamma=2(\xi x_0+\sigma_0+\omega t) \qquad \text{and} \qquad p\in R. \quad (2.48)$$

The evolution of the scattering data is then given by

$$\rho_t(p)=4\mathrm{i}p^2\rho+(\delta/a^2)J(\phi,\phi)+\mathrm{O}(\delta^2), \qquad p\in R \quad (2.49)$$

$$\xi_t=\delta\eta\Delta^{-1}\pi\exp(2\eta x_0-\tfrac{1}{2})|g(t)|\sin\gamma+\mathrm{O}(\delta^2) \quad (2.50)$$

$$\eta_t=\delta\xi^{-1}\eta^2\Delta^{-1}\pi\exp(2\eta x_0-\tfrac{1}{2})|g(t)|\sin\gamma+\mathrm{O}(\delta^2) \quad (2.51)$$

$$x_{0t}=-4\xi+\delta(B-A\epsilon)+\mathrm{O}(\delta^2) \quad (2.52)$$

$$\sigma_{0t}=2(\xi^2-\eta^2)-\tfrac{1}{2}\delta(2B\xi+A\epsilon\xi^{-1}(4\eta^2\xi^2x_0+\eta^2-2\xi^2))+\mathrm{O}(\delta^2), \quad (2.53)$$

where

$$A=\frac{\pi}{2\Delta^3}\mathrm{e}^{-1/2}|g|\sin\gamma\,\mathrm{e}^{2\eta x_0}, \qquad \text{and} \qquad B=\frac{1}{\pi}\mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{\rho I_p(\psi,\psi)\,\mathrm{d}\lambda'}{(\lambda'-\lambda)}\right). \quad (2.54)$$

A detailed analysis of the singular perturbation problem posed by equations (2.49)–(2.53) will be given elsewhere. Here we merely note that if we assume $\eta=\mathrm{O}(\delta)$ and $p^4\rho(t)=\mathrm{O}(\delta)$, so that we can then ignore the effects of the continuum to this order, the eigenvalue is constant in time. The principal effect is seen to be an imposed oscillatory motion on the velocity of the soliton (x_{0t}).

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